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Dual Geometric Field Theory[1]

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A Weyl geometry with a gauge-invariant, Riemannian subgeometry is used to geometrize the combined Einstein–Maxwell theory. A generalized Hamilton–Jacobi equation from particle mechanics emerges as an immediate consistency requirement. The time-independent, Coulomb field case is found to include at least lowest-order quantum effects as in wave mechanics. Possible microscopic entropy is identified.

1. INTRODUCTION

A straightforward generalization of existing Einstein–Maxwell theory into the gauge-invariant framework of the Weyl geometry (Weyl, 1922) is proposed. The broader kinematical framework of this geometry allows a geometric interpretation of the electromagnetic potential as well as the gravitational or metric field. However, the actual dynamical equations proposed by Weyl are not used; rather, a gauge-invariant metric tensor is constructed, and a Riemannian subgeometry is constructed on this. This subgeometry is gauge invariant, and allows an escape from the nonphysical effects of a pure Weyl geometry. It also allows the formulation of dynamical equations formally identical to those of the standard Einstein–Maxwell theory, but with two dimensionless constants. These equations will preclude the generation of further such subgeometries, leaving a dual-geometric description of the system. An immediate by-product of this is a scalar consistency relation, which the field quantities must satisfy also. This relation is a second-order, partial differential equation whose first-order term is of the form of the Hamilton–Jacobi equation of a charged particle in a combined gravitational and electromagnetic field. This will be identified as the equation of mechanics.

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The spherically symmetric solution to the Einstein–Maxwell equations is then examined. Spherical symmetry is not imposed on the new equation, though time independence will be assumed. Under these conditions, it separates into three ordinary differential equations that are further transformed into second-order linear equations. These will be seen to closely resemble the equations of the quantum theoretic, Coulomb-force problem. Indeed, the angular equations are identical. The radial equation is approximately the same, giving the same form in the nonrelativistic limit.

Closing sections then discuss constants, source motion, short-range effects, the physical identification of the new term in the mechanics equation in terms of entropy, and the concept of information as an exact, microscopic variable. The concept of "particles" is discussed also. Finally, the theory is contrasted with Weyl's original theory.

2. KINEMATICS—THE WEYL GEOMETRY, GAUGE INVARIANCE, AND A SUBGEOMETRY

The geometry proposed by Hermann Weyl (1923) as a framework for a unified field is defined by a metric tensor, $g_{\mu\nu}$, $g=\det(g_{\mu\nu})\neq 0$, and an intrinsic 4-vector, v_μ. Together these determine an affine connection

$$\Gamma^{\mu}_{\nu\alpha}=\left\{ {\mu \atop \nu\alpha} \right\}+\delta^{\mu}_{\nu}v_{\alpha}+\delta^{\mu}_{\alpha}v_{\nu}-g_{\nu\alpha}v^{\mu} \tag{1a}$$

where

$$\left\{ {\mu \atop \nu\alpha} \right\}=\tfrac{1}{2}g^{\mu\sigma}(g_{\nu\sigma,\alpha}+g_{\alpha\sigma,\nu}-g_{\nu\alpha,\sigma}) \tag{1b}$$

A comma before a subscript denotes the partial derivative with respect to a coordinate; that is, $,\alpha=\partial/\partial x^{\alpha}$.

The quantity $\Gamma^{\mu}_{\nu\alpha}$ is invariant under the conformal-gauge transformation

$$\bar{g}_{\mu\nu}=s(x^{\lambda})g_{\mu\nu}, \qquad s\neq 0 \tag{2a}$$

and

$$\bar{v}_{\mu}=v_{\mu}-\tfrac{1}{2}(\ln|s|)_{,\mu} \tag{2b}$$

We retain the absolute value for now, though it is common to restrict $s>0$. In the central-force example to be given, $s>0$ is used.

A quantity that is invariant under (2a) and (2b) is called gauge invariant because (2a) corresponds to a change of units or gauge. Clearly,

$$B^{\mu}_{\nu\alpha\gamma}=\Gamma^{\mu}_{\nu\gamma,\alpha}-\Gamma^{\mu}_{\nu\alpha,\gamma}+\Gamma^{\rho}_{\nu\gamma}\Gamma^{\mu}_{\rho\alpha}-\Gamma^{\rho}_{\nu\alpha}\,\Gamma^{\mu}_{\rho\gamma} \tag{3}$$

is also gauge invariant. This is the curvature tensor of the geometry.

The contracted tensor

$$B_{\nu\alpha}=B^{\mu}_{\nu\alpha\mu} \tag{4}$$

is gauge invariant also, and finally,

$$B=g^{\nu\alpha}B_{\nu\alpha} \tag{5}$$

is a scalar that transforms under (2a) as the inverse metric $g^{\mu\nu}$ does. This gives

$$\bar{B}=B/s \tag{6}$$

This allows us to define a gauge-invariant metric tensor

$$\hat{g}_{\mu\nu}=Bg_{\mu\nu} \tag{7}$$

if $B\neq 0$. Because this latter inequality is a gauge-invariant condition, we will assume for now that we can satisfy it.

Define

$$\left\{ {\hat{\mu} \atop \nu\alpha} \right\}=\tfrac{1}{2}\hat{g}^{\mu\sigma}(\hat{g}_{\nu\sigma,\alpha}+\hat{g}_{\alpha\sigma,\nu}-\hat{g}_{\nu\alpha,\sigma}) \tag{8}$$

We will assume that any raising or lowering of indices, or contracting on a hatted quantity, is always performed by $\hat{g}_{\mu\nu}$, whereas $g_{\mu\nu}$ is used for unhatted quantities. In general, a hat (ˆ) denotes a gauge-invariant quantity to be associated with $\hat{g}_{\mu\nu}$. The quantities $\Gamma^{\mu}_{\nu\alpha}$ and $B^{\mu}_{\nu\alpha\gamma}$ are not given hats, somewhat arbitrarily, because they predate the definition of $\hat{g}_{\mu\nu}$ and normally are not manipulated with it.

We define the covariant derivative using $\left\{ {\mu \atop \nu\alpha} \right\}$ by a semicolon (;), whereas the covariant derivative using $\left\{ {\hat{\mu} \atop \nu\alpha} \right\}$ is denoted by a double bar (||). In general, these uses with hatted and unhatted quantities should be reasonably clear from context.

Next, define the gauge-invariant Riemannian subgeometry through its curvature tensor

$$\hat{R}^{\mu}_{\nu\alpha\gamma}=\left\{\begin{matrix}\hat{\mu}\\ \nu\gamma\end{matrix}\right\}_{,\alpha}-\left\{\begin{matrix}\hat{\mu}\\ \nu\alpha\end{matrix}\right\}_{,\gamma}+\left\{\begin{matrix}\hat{\rho}\\ \nu\gamma\end{matrix}\right\}\left\{\begin{matrix}\hat{\mu}\\ \rho\alpha\end{matrix}\right\}-\left\{\begin{matrix}\hat{\rho}\\ \nu\alpha\end{matrix}\right\}\left\{\begin{matrix}\hat{\mu}\\ \rho\gamma\end{matrix}\right\} \tag{9}$$

We also have

$$\hat{R}_{\nu\alpha}=\hat{R}^{\mu}_{\nu\alpha\mu} \tag{10}$$

$$D=g^{\nu\alpha}\hat{R}_{\nu\alpha} \tag{11}$$

and

$$\hat{R}=\hat{g}^{\nu\alpha}\hat{R}_{\nu\alpha} \tag{12}$$

$\hat{R}$ is a true gauge invariant, while D transforms in the same manner as B. Finally, define

$$\hat{v}_{\mu}=v_{\mu}-\tfrac{1}{2}(\ln|B|)_{,\mu} \tag{13}$$

and

$$\hat{p}_{\mu\nu}=\hat{v}_{\nu,\mu}-\hat{v}_{\mu,\nu} \tag{14a}$$

$$=v_{\nu,\mu}-v_{\mu,\nu} \tag{14b}$$

We note that nothing has been said so far to prevent the formation of still another subgeometry using $Dg_{\mu\nu}$ instead of $Bg_{\mu\nu}$. This process could be carried on ad infinitum, provided no iteration is reached for which the new subgeometry has a scalar curvature of zero. This multiplicity of geometries would create unsatisfactory ambiguities if not eliminated.

3. DYNAMICS—GAUGE-INVARIANT EQUATIONS

We now identify $\hat{g}_{\mu\nu}$ with the Einstein, gravitational metric tensor. This differs from Weyl (Adler et al., 1965, pp. 401–417), and causes gravitation to be gauge invariant. We also identify the electromagnetic field tensor

$$\hat{f}_{\mu\nu}=j\hat{p}_{\mu\nu} \tag{15}$$

where j will be found to be a nontrivial dimensionless constant.

Next, assume an action of the form

$$I=\int\left[(\hat{R}-2l)-\tfrac{1}{2}\left(\hat{f}_{\mu\nu}\hat{f}^{\mu\nu}\right)\right](-\hat{g})^{1/2}\,d^4x \tag{16a}$$

l being a second dimensionless constant. We can rewrite this as

$$I=\int\left[(\hat{R}-2l)-\tfrac{1}{2}j^2\left(\hat{p}_{\mu\nu}\hat{p}^{\mu\nu}\right)\right](-\hat{g})^{1/2}\,d^4x \tag{16b}$$

Varying $\hat{g}_{\mu\nu}$ and $\hat{v}_\mu$, and equating the variational derivatives (Adler et al., 1965, pp. 324–328) to zero gives

$$\hat{R}_{\mu\nu}-l\hat{g}_{\mu\nu}=-j^2\left[\hat{p}_{\mu\sigma}\hat{p}^{\sigma}{}_{\nu}+\tfrac{1}{4}\hat{g}_{\mu\nu}\hat{p}_{\sigma\lambda}\hat{p}^{\sigma\lambda}\right] \tag{17}$$

and

$$\hat{p}^{\mu\nu}_{\parallel\nu}=0 \tag{18}$$

These, together with equations (14a) or (14b), give the Einstein–Maxwell theory, either with free fields or with singularities, as sources of $\hat{p}_{\mu\nu}$ (Adler et al., 1965, pp. 396–401). The use of the Riemannian subgeometry should bypass unphysical effects of the Weyl geometry (Adler et al., 1965, pp. 415–417).

To relate these more explicitly to ordinary units, first note that the formation of $\hat{g}_{\mu\nu}$ essentially causes any length measured with $\hat{g}_{\mu\nu}$ to be expressed in dimensionless units, using the scalar curvature B as a length standard (Weyl, 1922, p. 134). In other words,

$$B=1\ \text{natural unit}^{-2} \tag{19}$$

everywhere.

At this point, we simply assume that $B>0$, that our ordinary lab units correspond to some gauge

$$B=b=\text{const}>0 \tag{20}$$

and that the usual Einstein–Maxwell theory holds for that gauge. Then for $B=b$, assuming Gaussian units for the stress tensor (Jackson, 1962; Adler et al., 1965, pp. 261–280),

$$R_{\mu\nu}-\Lambda g_{\mu\nu}=-\frac{k}{4\pi}\left[F_{\mu\alpha}F^{\alpha}{}_{\nu}+\frac{1}{4}g_{\mu\nu}F_{\alpha\gamma}F^{\alpha\gamma}\right] \tag{21}$$

where Λ is the cosmological constant, and k is the Einstein gravitational constant,

$$k = \frac{8\pi G}{c^4} \tag{22}$$

G is Newton's gravitational constant, and c is the speed of light.

Now for $B = b$, we can show

$$R_{\mu\nu} = \hat{R}_{\mu\nu} \tag{23}$$

so

$$\hat{R}_{\mu\nu} - \left(\frac{\Lambda}{b}\right)\hat{g}_{\mu\nu} = -\frac{bk}{4\pi}\left[F_{\mu\alpha}\hat{g}^{\alpha\sigma}F_{\sigma\nu} + \frac{1}{4}\hat{g}_{\mu\nu}F_{\alpha\gamma}\hat{g}^{\alpha\rho}\hat{g}^{\gamma\lambda}F_{\rho\lambda}\right] \tag{24}$$

and we identify

$$\Lambda = bl \tag{25}$$

and

$$\hat{p}_{\mu\nu} = \frac{1}{2j}\left(\frac{bk}{\pi}\right)^{1/2} F_{\mu\nu} \tag{26}$$

The value of j remains separately important because v_ν already enters into $\Gamma^{\mu}_{\nu\alpha}$, and thus B, with a fixed relation to the metric tensor. Thus we need j to allow flexibility in the effect of its curl ($\hat{p}_{\mu\nu}$) on $\hat{g}_{\mu\nu}$ through the Einstein equations. If we juggle v_μ one place, the other will follow suit, so we cannot simply absorb j into v_μ or $\hat{p}_{\mu\nu}$. In fact, j^2 becomes a dimensionless gravitational constant.

We will defer a discussion of the value of b until later, except to note that the magnitudes of b and Λ *may* by comparable. This would leave l free to have a magnitude near unity.

Contracting equation (17) gives

$$\hat{R} = 4l \tag{27}$$

But

$$\begin{aligned}\hat{R} &= \hat{g}^{\nu\alpha}\hat{R}_{\nu\alpha} \\ &= \frac{g^{\nu\alpha}}{B}\hat{R}_{\nu\alpha} \\ &= \frac{D}{B}\end{aligned} \tag{28}$$

Thus the Riemannian quantity D and the Weyl quantity B always have the ratio $4l$. Because this is a constant, the dynamics have forced D always to be proportional to B in the same fixed ratio. A simple check then shows that $Dg_{\mu\nu}$ generates exactly the same $\hat{R}^{\mu}_{\nu\alpha\gamma}$ as the quantity $Bg_{\mu\nu}$ does. The infinite chain of subgeometries terminates after only one iteration. This is equivalent to requiring that our standard of length, B, be essentially unique. But equation (27) gives us this result by requiring D and B to be a consistent pair of standards.

Next, by direct calculation we can verify that

$$\Gamma^{\mu}_{\nu\alpha} = \left\{ \begin{matrix} \hat{\mu} \\ \nu\alpha \end{matrix} \right\} + \delta^{\mu}_{\nu}\hat{v}_{\alpha} + \delta^{\mu}_{\alpha}\hat{v}_{\nu} - \hat{g}_{\nu\alpha}\hat{v}^{\mu} \tag{29}$$

a relation of the same form as equation (1a), but using hatted quantities. Substituting this into equation (3) and performing a rather lengthy simplification, then contracting as in equation (4), and finally multiplying by $\hat{g}^{\nu\alpha}$ and contracting, we get

$$\hat{g}^{\nu\alpha}B_{\nu\alpha} = \hat{R} + 6\hat{v}^{\mu}_{\|\mu} + 6\hat{v}^{\mu}\hat{v}_{\mu} \tag{30}$$

But

$$\hat{g}^{\nu\alpha}B_{\nu\alpha} = \frac{g^{\nu\alpha}}{B}B_{\nu\alpha} = \frac{B}{B} = 1 \tag{31}$$

Then equation (27) implies immediately that

$$\hat{v}^{\mu}_{\|\mu} + \hat{v}^{\mu}\hat{v}_{\mu} = \frac{1-4l}{6} \tag{32}$$

This is a *necessary* relation among our various quantities if they are to fit consistently into the framework we have used. It is not simply a gauge condition, because it contains only gauge-invariant quantities.

Written out, equation (32) is

$$\frac{1}{(-\hat{g})^{1/2}}\left\{(-\hat{g})^{1/2}\hat{g}^{\mu\nu}\left[v_{\nu} - \left(\frac{1}{2}\ln B\right)_{,\nu}\right]\right\}_{,\mu}$$

$$+\hat{g}^{\mu\nu}\left[v_{\mu} - \left(\frac{1}{2}\ln B\right)_{,\mu}\right]\left[v_{\nu} - \left(\frac{1}{2}\ln B\right)_{,\nu}\right] = \frac{1-4l}{6} \tag{33a}$$

It is a second-order, partial differential equation for B, given v_{μ} and $\hat{g}_{\mu\nu}$.

Note that this equation actually completes the definition of the Weyl geometry. Equations (17), (18), and (14b) are actually solved for v_μ (in any convenient gauge) and $\hat{g}_{\mu\nu}$, not $g_{\mu\nu}$. The above then ties the gauge of B to that of v_μ, and gives B. This allows $g_{\mu\nu}$ to be calculated finally. Alternatively, one may choose $B=b$ from the beginning. Then one must still solve an equation of the form of equation (33a) to complete the determination of v_μ in that gauge. Either way, equation (33a) is necessary.

If we have solved equation (33a), we may then write equation (32) exactly in terms of first-order equations if we can find a function ϕ such that

$$(\ln\phi)_{,\mu}(-g)^{1/2}g^{\mu\nu}\left[v_\nu-\left(\tfrac{1}{2}\ln B\right)_{,\nu}\right]=-\left\{(-g)^{1/2}g^{\mu\nu}\left[v_\nu-\left(\tfrac{1}{2}\ln B\right)_{,\nu}\right]\right\}_{,\mu} \tag{33b}$$

Then ϕ is a gauge transformation that transforms (33a) into

$$\hat{g}^{\mu\nu}\left[\bar{v}_\mu+\left(\frac{1}{2}\ln\bar{B}\right)_{,\mu}\right]\left[\bar{v}_\nu-\left(\frac{1}{2}\ln\bar{B}\right)_{,\nu}\right]=\frac{1-4l}{6} \tag{33c}$$

in the new gauge. This is seen by expanding all the hatted quantities in (33a) into their Weyl components. One may then treat the first term as a product of B and the remaining quantities. By applying the product rule, we move the term generated by differentiating B into the $\hat{v}^\mu\hat{v}_\mu$ term. Equation (33b) then easily gives equation (33c).

Letting

$$S=\tfrac{1}{2}\ln B \tag{34}$$

we can write equation (33a) as

$$\hat{g}^{\mu\nu}(v_\mu-S_{,\mu})(v_\nu-S_{,\nu})=\frac{1-4l}{6}-\hat{v}^\mu_{\|\mu} \tag{35}$$

If $l<1/4$, then in a region in which

$$|\hat{v}^\mu_{\|\mu}|\ll\frac{1-4l}{6} \tag{36}$$

we have a quantity, S, which behaves like Hamilton's principle function for a charged *particle* in a combined electromagnetic and gravitational field (Goldstein, 1959; Landau and Lifshitz, 1962, pp. 148, 285). Of course these are all still field quantities. We have not explicitly introduced any particles, which makes this "particlelike" behavior all the more interesting. Accordingly, equation (32) is identified as the equation of mechanics. It should exhibit such "particlelike" behavior in regions for which equation (36) is true, and possibly also in regions in which equation (36) may not be true, but in which

$$\hat{v}^{\mu}_{\|\mu} \simeq \text{const} < \frac{1-4l}{6} \tag{37}$$

Also, we see immediately that, given a v_μ and $\hat{g}_{\mu\nu}$, a determination of B will be a problem at least as complex as the solution of the motion of a mechanical system, and, choosing a gauge such that $B=b$, is equally nontrivial, because it generates an equivalent problem. This is reminiscent of the situation in quantum theory where either the operators are stationary and the the state vector moves, or vice versa. Mixed cases are possible also. The parallel with quantum theory now will be seen to be even more precise in the case of a central force, to which we now turn.

4. THE CENTRALLY SYMMETRIC FIELD

The centrally symmetric, time-independent solution of the Einstein–Maxwell equations (Adler et al., 1965, pp. 396–401) is immediately applicable to equations (17), (18), and (14). It yields the Reissner–Nördstrom solution for a Coulomb field, charged point mass. This is

$$\hat{g}_{00}=e^{\nu}, \qquad \hat{g}_{11}=-e^{-\nu}, \qquad \hat{g}_{22}=-r^2$$

$$\hat{g}_{33}=-r^2\sin^2\theta, \qquad \hat{g}_{\mu\nu}=0, \qquad \mu\neq\nu \tag{38a}$$

and

$$v_0=-\frac{\delta}{j}-\frac{q}{jr}, \qquad v_i=0, \qquad i\geqslant 1 \tag{38b}$$

Also,

$$e^{\nu}=1-\frac{2m}{r}+\frac{q^2}{2r^2}-\frac{l}{3}r^2 \tag{39}$$

and δ, q, and m are constants. The choice of this form for v_μ effectively chooses the gauge, so we must regard equation (33a) as determining B.

Let

$$S=\tfrac{1}{2}\ln B \tag{40}$$

We can absorb the δ/j into S if S is a function of time (x^0), but we will assume time independence in what follows. Thus we leave the δ/j where it is. This is equivalent to omitting δ/j and assuming

$$S_{,0}=\frac{\delta}{j} \tag{41}$$

This is the same procedure one uses in Hamilton–Jacobi theory (Goldstein, 1959) to separate the time, so δ/j plays the role of a total "energy."

A straightforward calculation now gives

$$\frac{\left(r^2e^{\nu}S_{,1}\right)_{,1}}{r^2}+\frac{e^{-\nu}}{j^2r^2}(q+\delta r)^2-e^{\nu}(S_{,1})^2$$
$$+\frac{1}{r^2}\left\{\frac{(S_{,2}\sin\theta)_{,2}}{\sin\theta}-(S_{,2})^2+\frac{1}{\sin^2\theta}\left[S_{,3,3}-(S_{,3})^2\right]\right\}=\frac{1-4l}{6} \tag{42}$$

from equation (32). We have assumed time independence, but not spherical symmetry for S. There seems to be no justification for forcing spherical symmetry on S, because it cancels from both $\hat{p}_{\mu\nu}$ and $\hat{g}_{\mu\nu}$, leaving them spherically symmetric.

Equation (42) now neatly separates, assuming as in Hamilton–Jacobi theory,

$$S=U(r)+T(\theta)+A(\phi) \tag{43}$$

This gives

$$A''-(A')^2=n_1^2 \tag{44}$$

$$T''+T'\cot\theta-(T')^2+\frac{n_1^2}{\sin^2\theta}=n_2(n_2+1) \tag{45}$$

and

$$(r^2e^{\nu}U')'-r^2e^{\nu}(U')^2+\frac{e^{-\nu}}{j^2}(q+\delta r)^2+n_2(n_2+1)=\frac{1-4l}{6}r^2 \tag{46}$$

The quantities n_1 and n_2 form separation constants whose values we anticipate to be integral. To see this, note that all of the above equations are Ricatti equations (Sokolnikoff and Redheffer, 1966). They become linear under

$$f' = -h'/h \tag{47}$$

This gives

$$A' = -y'/y \tag{48a}$$

$$y'' + n_1^2 y = 0 \tag{48b}$$

$$T' = -P'/P \tag{49a}$$

$$P'' + \cot\theta P' + \left[n_2(n_2+1) - \frac{n_1^2}{\sin^2\theta} \right] P = 0 \tag{49b}$$

$$U' = -u'/u \tag{50a}$$

and

$$(r^2 e^{\nu} u')' + \left[\frac{1-4l}{6} r^2 - \frac{e^{-\nu}}{j^2}(q+\delta r)^2 - n_2(n_2+1) \right] u = 0 \tag{50b}$$

Equations (48b) and (49b) are precisely the angular equations from the central-force problem in quantum theory. Provided we require y and P to be single valued and well behaved they give n_1 and n_2 as integers:

$$n_2 \geqslant 0 \tag{51}$$

and

$$-n_2 \leqslant n_1 \leqslant n_2 \tag{52}$$

But clearly y must be single valued, in contrast to Hamilton–Jacobi theory. Thus n_1 is integral to ensure that our fields and gauge are single-valued functions of ϕ. The requirements on P are not so obvious, and to try to see them we note

$$\begin{aligned} S &= U + T + A \\ &= -\ln|u| - \ln|P| - \ln|y| \\ &= \frac{1}{2} \ln \frac{1}{u^2 P^2 y^2} \end{aligned} \tag{53}$$

This gives

$$B=(uPy)^{-2} \tag{54}$$

Conveniently enough, this gives $B>0$ provided u, P, and y are never infinite. On the other hand, B will be infinite for some values of the coordinates; that is, wherever any of the three functions has a zero. If we require B to always be finite and nonzero, the only allowed eigenvalues are $n_1=0$ and $n_2=0$.

However, we can note a correspondence between the regions for which B would be infinite and the regions in quantum theory for which $\psi^*\psi=0$, the regions where a particle is never found. Here, we have $u^2P^2y^2=0$ instead, and we have singular surfaces of B instead of vanishing probabilities. By equation (7), the Weyl metric vanishes on these surfaces. We will proceed at this time with equations (51) and (52).

Now, to discuss the radial equation, equation (50b), we need to complete the transformation to ordinary lab units. To do this, let the old coordinates be barred and the new ones be unbarred. Then let

$$r=\bar{r}/\sqrt{b} \tag{55a}$$

and

$$x^0=\bar{x}^0/\sqrt{b} \tag{55b}$$

Then note that we have

$$d\hat{s}^2=e^{\nu}(d\bar{x}^0)^2-e^{-\nu}(d\bar{r})^2-\bar{r}^2(d\theta)^2-\bar{r}^2\sin^2\theta(d\phi)^2 \tag{56}$$

For the new coordinates,

$$d\hat{s}^2=b\left[e^{\nu}(dx^0)^2-e^{-\nu}(dr)^2-r^2(d\theta)^2-r^2\sin^2\theta(d\phi)^2\right] \tag{57}$$

where

$$e^{\nu}=1-\frac{2m}{\sqrt{b}\,r}+\frac{q^2}{2br^2}-\frac{bl}{3}r^2 \tag{58}$$

This actually achieves the gauge $B=b$ explicitly. It gives for equation (50b),

$$(r^2e^{\nu}u')'+\left[\frac{b(1-4l)}{6}r^2-\frac{e^{-\nu}}{j^2}\left(q+\sqrt{b}\,\delta r\right)^2-n_2(n_2+1)\right]u=0 \tag{59}$$

In regions in which

$$e^{\nu}\simeq 1 \tag{60}$$

this gives a form similar to the relativistic Schrödinger, radial equation (Schiff, 1949),

$$(r^2u')'+\left[\frac{b(1-4l)}{6}r^2-\frac{1}{j^2}\left(q+\sqrt{b}\,\delta r\right)^2-n_2(n_2+1)\right]u=0 \tag{61}$$

Thus, e^{ν} appears as a general relativistic correction on the radial equation for very small r, and very large r.

For very small r,

$$e^{\nu}\simeq 1-\frac{2m}{\sqrt{b}\,r}+\frac{q^2}{2br^2} \tag{62}$$

whereas for very large r,

$$e^{\nu}\simeq 1-\frac{bl}{3}r^2 \tag{63}$$

5. THE CONSTANTS AND THEIR VALUES

Equation (61) resembles the relativistic Schrödinger, radial equation, but appears to have two signs reversed. Assuming $l<1/4$, we see that the first and second terms in the bracket enter with reversed signs. Nevertheless, it gives the correct, nonrelativistic, radial form as a limit.

To see this, consider that equation (61) can be written (Schiff, 1949)

$$\left[-\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)+\frac{n_2(n_2+1)}{r^2}\right]u=-b\left[\left(\frac{\delta}{j}+\frac{q}{j\sqrt{b}\,r}\right)^2-\frac{1-4l}{6}\right]u \tag{64}$$

Define

$$\frac{\delta'}{j}=\frac{\delta}{j}-\left(\frac{1-4l}{6}\right)^{1/2} \tag{65}$$

Assume

$$\left|\frac{\delta'}{j}\right|\ll\left(\frac{1-4l}{6}\right)^{1/2} \tag{66}$$

and

$$\left|\frac{q}{j\sqrt{b}\,r}\right| \ll \left(\frac{1-4l}{6}\right)^{1/2} \tag{67}$$

Then equation (64) becomes

$$\left[-\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)+\frac{n_2(n_2+1)}{r^2}\right]u \simeq -2b\left(\frac{1-4l}{6}\right)^{1/2}\left(\frac{\delta'}{j}+\frac{q}{j\sqrt{b}\,r}\right)u \tag{68}$$

With a few immediately suggested identifications, this has precisely the form of the radial equation for the Schrödinger treatment of the nonrelativistic, one-electron atom (Schiff, 1949).

We tentatively identify

$$-\frac{2b\delta'}{j}\left(\frac{1-4l}{6}\right)^{1/2}=\frac{2m_eE'}{\hbar^2} \tag{69}$$

and

$$-\frac{2\sqrt{b}\,q}{j}\left(\frac{1-4l}{6}\right)^{1/2}=\frac{2m_eZe^2}{\hbar^2} \tag{70}$$

where e is the magnitude of the electronic charge in Gaussian units, m_e is the electronic mass, E' is the Schrödinger energy, Z is the atomic number, and $\hbar$ is Planck's constant over 2π. The form of the Reissner–Nördstrom metric[3] suggests the further identification of

$$\frac{q^2}{2b}=\frac{G(Ze)^2}{c^4} \tag{71}$$

These give

$$q=(2bG)^{1/2}\frac{Ze}{c^2} \tag{72}$$

$$\delta'=\frac{(2G)^{1/2}}{ec^2}E' \tag{73}$$

[3]See Adler et al., 1965, p. 401 (but use Gaussian units).

and

$$j = -\frac{\hbar^2(2G)^{1/2}}{ec^2}\frac{b}{m_e}\left(\frac{1-4l}{6}\right)^{1/2} \tag{74}$$

Note that neither l nor b has yet received any suggested values.

5.1. The Gauge Constant. The gauge constant relates the natural marker system defined by the scalar B to our ordinary lab units. Note that b has appeared in analogy to the electronic mass in Schrödinger theory. In such a position it would affect the wavelengths of the spectral times that one may use for lab standards. Indeed, equation (74) gives

$$b \sim m_e \tag{75}$$

A closer comparison with relativistic Schrödinger theory suggests

$$b = \frac{6}{1-4l}\frac{m_e^2c^2}{\hbar^2} \tag{76}$$

Thus, it appears possible that b may easily be correlated directly with laboratory units if l is known or $|4l| \ll 1$.

Further speculation on this suggests the possibility that structures such as muonic atoms may be associated with different determinations of b. This would be somewhat similar to the association of different Bohr radii with different atomic energy levels (Margenau and Murphy, 1962, p. 367). In other words, muonic atoms would somehow represent excited gauge states. This would seem to require that the theory predict some eigenvalue spectrum for b when equation (59) or its counterpart in a given solution is considered exactly. This has not been demonstrated. It would also require that such different determinations of b can be handled consistently. This would require that b could be treated as a system constant such as total energy, rather than a universal constant such as (presumably) the fine structure constant. This also has not been explored.

If $l \neq 0$ and b is not a universal constant, neither is Λ.

5.2. The Remaining Constants. Note that the quantity Ze actually canceled from j. The e in the denominator is associated directly with the constant, and illustrates how only one charged singularity might appear mathematically to be two interacting charges. Closer comparison to relativistic quantum theory suggests

$$\frac{q}{j} = -Z\frac{e^2}{\hbar c} \sim e^2$$

and not just e. We also have then for j, the more accurate expression

$$j = -2\left(\frac{3}{1-4l}\right)^{1/2} \frac{\sqrt{G}\, m_e}{e} \tag{77}$$

The constant j depends explicitly on the value of k in equation (22). Note that k only enters the metric through the electromagnetic term. Because the $2m/r$ term is separately adjusted to the Newtonian limit, the usual arguments concerning the value of k may be less valid (Adler et al., 1965, pp. 173, 277–280, 400). Furthermore, equation (77) suggests that the product Gm_e^2 is constant, suggesting that G possibly may be smaller for muons in muonic atoms or for similar systems.

The dimensionless constant

$$l = \frac{1}{4}\frac{D}{B} \tag{78}$$

As the ratio of these two scalar curvatures, Riemannian over Weyl, it represents the correlation between two possible standards of length. In other words,

$$\frac{Dg_{\mu\nu}}{Bg_{\mu\nu}} = 4l$$

The two possible forms for the gauge-invariant metric always have this fixed ratio, like feet and meters. Its value must be constant to keep the system consistent.

As such, l represents a very fundamental property of the entire dual-geometric model. No value is yet proposed, but the two values

$$l = 1/4 \tag{79}$$

and

$$l = 0 \tag{80}$$

would seem to be the least interesting choices. Both give trivial limits for the dual standard of length in which the duality is somewhat masked. But it seems plausible that it is the overall *duality* that generates any semblance of quantum equations in this framework. In other words, we still have a dualistic, complementary description of the world. Now it is through dual geometries. Furthermore, if l is an extreme number such as 10^{-80}, it might tend to naturally divide the Weyl and Riemannian geometries into a geometry of microcosm and a geometry of macroscosm. Additionally equation (79) would lead to singular values in many of the preceding equations.

As for m, its value should simply be (Adler et al., 1965, p. 400)

$$m = \frac{2GM\sqrt{b}}{c^2} \tag{81}$$

where M is the singularity mass in ordinary units. Note that the $\sqrt{b}$ will cancel in (58). We do not need it for e^{ν}.

The constant δ presumably has eigenvalues determined directly from the radial equation. We identified δ/j with "energy" in equations (41), (65), and (73). It is worthwhile noting that negative "energy" works as well by reversing the sign of δ/j in equations (38b) and (41), and corresponding signs in succeeding equations. This leads to sign reversals in equations (70) and (73). It would eventually reverse the sign of j, though not j^2, of course.

6. DISCUSSION

6.1. Source Motion. The consistency relation, equation (32), has so far been identified in this paper as the mechanical equation. This is based on immediate similarities to forms from classical and quantum mechanics; however, we also have the intuitive feeling that mechanics in physics should be relevant to the motion of field sources, especially if they are singularities. Because the motions described in equation (32) are those of fields, there is by no means yet any *direct* link with the motions of the field sources. Equation (32) represents "mechanics" by analogy. Its behavior is mechanical; that is, it contains quantities that might be perceived as mechanical simply because they behave as we expect mechanical systems to behave. This results from the identities, equations (30) and (31), and the dynamical condition, equation (27). Thus it fulfills most of Eddington's criteria for identification of quantities and laws (Eddington, 1965, p. 222).

But then, how do the field sources actually move? This question remains relevant. There seem to be several possible approaches to an answer.

If we retain singular sources, then the works of Infeld, Rohrlich, Dirac, Einstein, and many others, (Infeld and Plebanski, 1960; Rohrlich, 1965) all suggest that singularities must basically move according to the accepted classical laws (using the subgeometry). These seem to follow from the Einstein equations and conservation laws. Such classical motions would raise stability questions for multisingularity systems. The same problems arise in the Bohr theory of the atom. Radiation damping may cause system decay. Possible solutions might include negative masses (Synge, 1960) or use of mixtures of advanced and retarded fields to inhibit radiation (Infeld and Wallace, 1940). Or, stable, ground state systems might appear as only one singularity, but excited states might involve more than one.

Alternatively, to directly correlate equation (32) to singularity motion, it would seem necessary to demonstrate the consistency of any such correlation with the field equations. This has not been done in this paper. Indeed, the above-mentioned works suggest that it might be impossible because of the second-order term in equation (32). This would not seem to exclude some demonstration of an average or statistical correlation between equation (32) and singularity motion. Furthermore, behavior restrictions on B, and the nonlinearity of equation (32) may produce new restrictions on singularity motion.

Then what of nonsingular sources? One possibility is that of Wheeler's "wormholes." (Adler et al., 1975, pp. 507–532). One requires the free-field Maxwell equations to be true everywhere, and endows space-time with novel topologies.

Another possibility is one previously suggested by the author (Rankin, 1971). It assumes that electromagnetic sources move microscopically so that the total Lorentz force density vanishes at each point in space-time. An example of such a system is given by the action integral

$$I=\int\left\{(\hat{R}-2l)-\tfrac{1}{2}j^2\left[\hat{p}_{\mu\nu}\hat{p}^{\mu\nu}+(\hat{u}^{\alpha}\hat{u}_{\alpha})\hat{p}_{\mu\nu}{}^{*}\hat{p}^{\mu\nu}\right]\right\}(-\hat{g})^{1/2}\,d^4x \qquad (82)$$

This is considered to be a functional of $\hat{g}_{\mu\nu}$, $\hat{v}_{\mu}$, and $\hat{u}_{\alpha}$. The quantity

$${}^{*}\hat{p}^{\mu\nu}=\frac{1}{2(-\hat{g})^{1/2}}\varepsilon^{\mu\nu\alpha\beta}\hat{p}_{\alpha\beta} \qquad (83)$$

is the dual of $\hat{p}_{\mu\nu}$. One disadvantage is the lack of an immediate geometric identification for $\hat{u}_{\alpha}$. The system also appears considerably more complex mathematically than the theory with singular sources.

In view of all of the above, the correlation between equation (32) and our more intuitive notions of mechanics remains incomplete. The potential for two separate concepts of mechanics is present. One would involve equation (32) and the other would refer directly to source motion. Such a dualism should be self-consistent because different quantities are involved in the separate concepts. However, a true understanding of time-dependent phenomena will require a detailed correlation of concepts. Questions about electromagnetic radiation during "energy" transitions in equation (32) provide an example. It would seem that such questions may be answered from the formalism, at least in principle. If so, links between the two "mechanics" may not have to be supplied by postulates.

6.2. Short-Range Forces and Other Radial Effects. So far, the results of equation (32) can be seen to be analogous to nonrelativistic quantum theory in certain limits. Any immediate short-range effects obviously must involve the complete, general relativistic structure. Without attempting such detailed analysis, we may still make some rough observations.

Using the spherically symmetric case as a guide, we see that metrical corrections will enter the radial equation twice. We have from equations (61) and (59),

$$q \to q e^{-\nu/2} \tag{84a}$$

and

$$\delta \to \delta e^{-\nu/2} \tag{84b}$$

as corrections to the potential or Coulomb force terms. But we also see

$$r^2 \to r^2 e^{\nu} \tag{84c}$$

in the derivative term.

Using accepted nucleon and nuclear parameters (Evans, 1965), equations (62), (72), and (81) would indicate the well-known result that general relativistic corrections to atomic and nuclear problems become significant for very small r values, on the order of 10^{-34} cm. The electrical term in the metric becomes significant first, because the mass term approaches unity only for r of the order of 10^{-53} cm. The radial marker r is not actually the radius (Landau and Lifshitz, 1962, pp. 272–274), which is instead given by

$$\rho = \int_0^r e^{-\nu/2}\, dr \tag{85}$$

provided there is no Schwarzchild singularity in the integration range. For nuclear and subnuclear systems there is no singularity (Adler et al., 1965, p. 401) and differences between ρ and r are negligible for $r \gtrsim 10^{-32}$ cm.

Thus general relativistic effects would not appear responsible for any short-range forces at the usual nuclear and nucleon range of about 10^{-13} cm. For truly singular particles, however, corrections would be expected to become quite important at the much smaller radii indicated above. Their primary effects might conceivably be the modifications introduced into the boundary value problem at a radius of zero or at a singular surface, if one exists.

The outer r limit also should place conditions on solutions. It may be interesting to see if either limit, or both together, require sufficient restrictions to force eigenvalues on more general source parameters such as q or m, in addition to δ.

One other possible effect may be of interest for short-range and exchange reactions in this model. If we ignore general relativity and simply examine the approximate form of equation (32) for two charged singularities, we see that it may be similar to the hydrogen, molecular ion problem in quantum mechanics (Margenau and Murphy, 1962, pp. 385–387). Thus the two-singularity problem may appear like a quantum, three-body problem. This may introduce some exchange reaction effects in a basic way.

6.3. Entropy and Information. The quantity

$$S_m = -\tfrac{1}{2}\ln(B/b) \tag{86}$$

at any point of space-time may be compared to the information capacity of a system of N states (Raisbeck, 1964) given by

$$\log_2 N \tag{87}$$

We have seen that $(1/B)^{1/2}$ is analogous to the quantum state function. Furthermore, the solution of the problem of system motion allows us to set

$$B = b \tag{88}$$

or

$$S_m = 0 \tag{89}$$

By solving the problem, we have "extracted" the information. This suggests we define (86) as the "mechanical information field." It is not gauge invariant, however, and may be formally zeroed even without solving for the system motion.

The gauge-invariant generalization of this concept seems to be the current

$$\hat{v}_\mu = v_\mu + (S_m)_{,\mu} \tag{90}$$

But Tolman (1962) states the general relativistic form of the second law of thermodynamics as

$$\left[(-g)^{1/2} S^\mu\right]_{,\mu} \delta x^1 \delta x^2 \delta x^3 \delta x^0 \geqslant \frac{\delta Q_0}{T_0} \tag{91}$$

where S^μ is the entropy vector. We can rewrite equation (91) in our subgeometry as

$$\left[(-\hat{g})^{1/2} \hat{S}^\mu\right]_{,\mu} \delta x^1 \delta x^2 \delta x^3 \delta x^0 \geqslant \frac{\delta Q_0}{T_0} \tag{92}$$

This, with equations (86) and (87), suggests that we tentatively identify

$$\hat{v}^\mu = \text{entropy vector} \tag{93}$$

Then equation (32) gives

$$\left[(-\hat{g})^{1/2} \hat{v}^\mu\right]_{,\mu} \delta x^1 \delta x^2 \delta x^3 \delta x^0 = (-\hat{g})^{1/2} \left(\frac{1-4l}{6} - \hat{v}^\mu \hat{v}_\mu \right) \delta x^1 \delta x^2 \delta x^3 \delta x^0 \tag{94}$$

as a possible microscopic statement of the second law of thermodynamics. In general, the current $\hat{v}^\nu$ would not represent a conserved quantity unless behavior is purely classical. This occurs when

$$\hat{v}^\mu \hat{v}_\mu = \frac{1-4l}{6} \tag{95}$$

If these simple considerations are verified, they may show a microscopic tendency toward irreversibility.

6.4. On Particles. Except for the pure singularity at $r = 0$, there seem to be no "particles" in the usual sense in the solution used. However, the agreement of equation form with quantum theory suggests that our model may be dealing with the same entities as wave mechanics. This suggests that either (1) we are dealing with totally ionized and isolated cores, or (2) individual particles lose their identity in the atom and the nucleus, and only the total charge and mass have the usual meaning.

The difficulty with (2) is that it would presumably require truly neutral systems, such as complete atoms, to have $q=0$ and $m\neq 0$. These would contain Schwarzchild singularities.

Although such solutions may be found, any parallel between equation (68) and conventional atomic physics would be for $q\neq 0$. This suggests that the electronic wave functions are more a property of the central field than of orbiting particles. This seems analogous to the one-body, central-field problem in celestial mechanics. In that instance, the central-field predetermines the geometric geodesics available for satellites. The mutual interactions of the satellites, and the satellite fields are ignored in such a treatment.

This suggests that more general solutions may show genuine electronic charge distributions away from $r=0$. However, one may believe that all electromagnetic sources are true singularities. If additional particles must appear as such additional singularities, they may present difficulties for the mathematical techniques of general relativity. But, nonsingular sources require a broader formalism (Rankin, 1971) because of equation (18).

The absence of dipole magnetic fields on the core is another limitation. Presumably a spinning, singular source can give some relief. The use of the Reissner–Nördstrom metric should have excluded such systems from our solution at the outset. However, some adaptation of a Kerr–Newman solution (Adler et al., 1975, Chapters 7 and 15) should alleviate this.

One promising feature of the model pertains directly to the Reissner–Nördstrom metric. As Weyl notes, there are no renormalization problems like those that plague special relativistic point-charge theories (Weyl, 1922, pp. 260–273). The mass is simply m.

Photonlike effects are not yet evident in this model. Some form of time-dependent solution may be required to verify their presence, or absence, in radiation emission, absorption, or propagation. However, note that equation (32) continues to apply, even to free radiation fields in the absence of sources.

7. CONTRAST WITH WEYL'S THEORY

Some closing comparisons with the original Weyl theory may now be of interest. Adapting Eddington's analysis (Eddington, 1965, pp. 206–212) to the notation of this paper gives Weyl's equations as

$$B=b=\text{const} \tag{96a}$$

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\frac{b}{4}g_{\mu\nu}=-\frac{j^2}{b}\left(p_{\mu\sigma}p^{\sigma}{}_{\nu}+\frac{1}{4}g_{\mu\nu}p_{\sigma\alpha}p^{\sigma\alpha}\right)-6\left(v_\mu v_\nu-\frac{1}{2}g_{\mu\nu}v^\alpha v_\alpha\right) \tag{97a}$$

$$p^{\mu\nu}_{;\nu}=-\frac{6b}{j^2}v^\mu \tag{98a}$$

$$p_{\mu\nu}=v_{\nu,\mu}-v_{\mu,\nu} \tag{99a}$$

$$v^\mu_{;\mu}=0 \tag{100a}$$

By contrast, the equations of this paper written in the constant gauge are

$$B=b=\text{const} \tag{96b}$$

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+blg_{\mu\nu}=-\frac{j^2}{b}\left(p_{\mu\sigma}p^{\sigma}{}_{\nu}+\frac{1}{4}g_{\mu\nu}p_{\sigma\alpha}p^{\sigma\alpha}\right) \tag{97b}$$

$$p^{\mu\nu}_{;\nu}=0 \tag{98b}$$

$$p_{\mu\nu}=v_{\nu,\mu}-v_{\mu,\nu} \tag{99b}$$

and

$$v^\mu_{;\mu}+v^\mu v_\mu=\frac{b(1-4l)}{6} \tag{100b}$$

Both sets have purposely been written in terms of the original, unhatted variables using the unhatted Christoffel symbols for the covariant derivatives.

The greatest difference is clearly between equations (98a) and (98b). Weyl has continuous sources present everywhere the potential is nonzero. Equation (98b) represents electromagnetic fields with singularities as sources. Because the presence, or absence, of sources is a gauge-invariant condition, a clear distinction is established. Beyond that, equation (97b) is the familiar Einstein equation with only the standard, electromagnetic stress tensor on the right side. Equation (97a) contains additional stress terms, and also seems to indicate a definite l value of $1/4$. Finally, Weyl's constant gauge corresponds to an electromagnetic Lorentz gauge, equation (100a). Equation (100b), as seen earlier, represents a considerably more complicated condition, leading to forms familiar from mechanics in certain limits. Its deceptively simple appearance here as a gauge condition masks those results. On the other hand, this form provides evidence that the addition of equation (32) to the other field equations does not overdetermine the system mathematically.

The two theories will coincide for the special case of

$$l=1/4 \tag{101}$$

and

$$v^{\mu}=0 \tag{102}$$

Finally, we can put the original Weyl theory into an action principle of the form used in this paper as

$$I_W=\int\left[\left(\hat{R}-\tfrac{1}{2}\right)-\tfrac{1}{2}j^2\left(\hat{p}_{\mu\nu}\hat{p}^{\mu\nu}\right)+6\hat{v}_{\alpha}\hat{v}^{\alpha}\right](-\hat{g})^{1/2}\,d^4x \tag{103}$$

Treating this as a functional of $\hat{g}_{\mu\nu}$ and $\hat{v}_{\mu}$, and using equations (30) and (31), then gives equations (96a)–(100a) for the gauge $B=b$. We also find

$$\hat{R}=1-6\hat{v}_{\alpha}\hat{v}^{\alpha} \tag{104}$$

Because this is not generally constant, more than one subgeometry can be generated by self-gauging against scalar curvatures such as B and D. This does not seem serious mathematically. However, it raises questions concerning uniqueness of the standards of length obtained by such self-gauging.

From a physical viewpoint, such uniqueness, as expressed by equation (27), might be required as a postulate. That would effectively reduce equation (32) itself to a postulate.

Also, equation (104) implies

$$\hat{v}^{\mu}{}_{\|\mu}=0 \tag{105}$$

This equation would not show the similarities to particle mechanics found in equation (32). It would also make it impossible to interpret $\hat{v}_{\mu}$ in terms of entropy because it implies strict conservation, a property not shared with entropy.

8. CONCLUSIONS

A gauge-invariant form of the Einstein–Maxwell theory has been derived. The model has been imbedded in a self-gauging, Weyl geometry with a Riemannian subgeometry. This has given an additional consistency relation resembling an equation from particle mechanics. For central symmetry, it has led, in the nonrelativistic limit, to forms familiar from quantum theory. Quantization has resulted naturally from this consistency relation plus boundary conditions and restrictions on behavior of the Weyl scalar curvature. The nonvanishing of this curvature has been the primary condition. Both the Schrödinger wave function and the thermodynamic entropy vector have been suggested as having geometric interpretations in this model. The original Weyl theory has also been framed in the notation of this paper. It has been seen to differ with the theory of this paper in several ways.

Several other points have been discussed. Among these have been source motion, short-range effects, and some possible implications for particle theory. Major questions concerning these areas have remained unanswered. These included correlation between source motion and the "mechanics" of the consistency relation, and the detailed effect of general relativity on eigenvalue spectra generated by the consistency relation. Solutions with spin (Kerr–Newman) have not been examined. Possible portrayal of some atomic and particle phenomena as "excited" gauge states has been suggested, but not demonstrated. The possibility that "third body" exchange effects may occur naturally for two bodies in this theory has been mentioned.

9. NOTATION

Other than as noted in the paper's body, the following apply:

1. Coordinates are numbered from 0 to 3, with $x^0 = ct$ and x^1, x^2, and x^3 as space coordinates.

2. The metric tensor signature is $+---$.

3. All electromagnetic potentials are correlated with flat space-time forms by

$$A_\mu = (\Phi, -\mathbf{A})$$

whereas the field tensors are defined by

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$$

This is opposite that of many references.

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